DEFORMATION AND APPLICABILITY OF SURFACES IN LIE SPHERE GEOMETRY

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ABSTRACT. The theory of surfaces in Euclidean space can be naturally formulated in the more general context of Legendre surfaces into the space of contact elements. We address the question of deformability of Legendre surfaces with respect to the symmetry group of Lie sphere contact transformations from the point of view of the deformation theory of submanifolds in homogeneous spaces. Necessary and sufficient conditions are provided for a Legendre surface to admit non-trivial deformations and the corresponding existence problem is discussed.

INTRODUCTION

The classical problems of applicability of surfaces in Euclidean, projective and conformal geometry fit into the general theory of deformation of submanifolds in homogeneous spaces as formulated by É. Cartan [6, 7] and further developed by P. Griffiths and G. Jensen [14, 15]. Two submanifolds in a homogeneous space, $f, \tilde{f} : M \to G/K$, are k-th order deformations of each other if there exists a smooth map $v : M \to G$ such that, for every $p \in M$, the Taylor expansions about p of $\tilde{f}$ and $v(p) \cdot f$ agree through k-th order terms; if v is constant, f and $\tilde{f}$ are congruent with respect to G. Of course, for each concrete geometric situation there is a specific problem to solve. In Euclidean space, two surfaces are applicable in Gauss' sense if they are first order Euclidean deformations of each other, which means they are isometric with respect to the induced metrics, and are congruent to second order. In projective 3-space, Fubini's notion of applicability of surfaces goes to second order and rigidity to third order. In Möbius and Laguerre geometry, second order deformable surfaces coincide with isothermic and L-isothermic surfaces, respectively [17, 19].

This paper studies the deformation problem for surfaces in another classical geometry: Lie sphere geometry. It is the outcome of our attempts to understand Lie-applicable surfaces within the general theory of deformation. Lie-applicable surfaces were considered by Blaschke and his collaborators in the 1920s [4]. Recently, interest in Lie-applicable surfaces has reappeared in the work of Ferapontov concerning the relation between Lie sphere geometry of hypersurfaces and the theory of integrable systems [9, 11].

To put our discussion in perspective let us recall some facts about Lie sphere geometry. Any smooth immersion of an oriented surface into 3-space has a contact lift to the unit sphere bundle Λ of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ called the Legendre lift. The unit sphere bundle is acted on transitively by the group of Lie sphere transformations. This group is the group of contact transformations generated by the conformal transformations of S^3 together with the group of normal shifts, which transform an oriented surface to its parallel surface at a fixed oriented distance in the normal direction. Thus the Lie sphere group acts on the set of Legendre surfaces. The principal aim of Lie sphere geometry is to study the properties of an immersion which are invariant under this action [4, 8, 23]. Blaschke associated with any Legendre surface a canonical coframe $\mathcal{C}$ and proved that the position of a generic Legendre surface is completely determined by its canonical coframe, up to Lie sphere transformations. However, he observed that there are exceptions to this rigidity result. Accordingly, two non-congruent Legendre immersions $f, \widetilde{f}$ are called *Lie-applicable* if $\mathcal{C} = \widetilde{\mathcal{C}}$. As we will see in Section 2, examples of Lie-applicable surfaces include the Legendrian lifts of isothermic and L-isothermic surfaces, which are known to constitute integrable systems [5, 2, 21].

In the paper, we think of Λ as a homogeneous space of the identity component $\widetilde{G}$ of the Lie sphere group. We will prove that two (nondegenerate) Legendre immersions are Lie-applicable if and only if they are second order deformations of each other; that two Legendre immersions are always local first order deformations of each other; and that they have third order rigidity. Further, we show how to recognize that a Legendre surface is deformable and in this case how to find all its deformations.

In Section 1, we collect some background material about the Lie sphere geometry of surfaces and develop the method of moving frames in this context (see [4, 8]). We identify Λ with the space of isotropic 2-spaces in $\mathbb{R}^{4,2}$ and linearize the action of Lie transformations. In this model, the identity component of O(4, 2) acts on Λ by contact diffeomorphisms and provides a double cover of $\widetilde{G}$. We then apply the method of moving frames to study Legendre surfaces and recall the construction of a canonical lift to the group $\widetilde{G}$ under a natural nondegeneracy assumption. For any Legendre surface, we introduce the canonical coframe (α^1, α^2), which turns out to be the analogue of that considered by Blaschke [4], and the quadratic and cubic forms of the surface. We define a set of local differential invariants $q_1, q_2, p_1, p_2, r_1, r_2$ for a Legendre surface and relate them to the classical ones discussed by Blaschke and Ferapontov. We then deduce the compatibility conditions, which play the role of the Gauss–Codazzi equations for a Legendre surface in Lie sphere geometry. The functions q_1, q_2, p_1, p_2 are completely determined by the canonical coframe, while r_1, r_2 govern the extrinsic geometry of the Legendre immersion.

In Section 2, we investigate the class of Legendre surfaces which are not determined by the canonical coframe. We take the point of view of the deformation theory of submanifolds in homogeneous spaces. We introduce the concept of deformation and discuss the related questions of analytic contact and applicability. We study the problem of second order deformation of Legendre immersions and prove that two nondegenerate Legendre immersions are second order deformations of each other precisely when they are Lie-applicable, or equivalently, when they have the same quotient of cubic to quadratic forms (see Theorem 2.10).

In Section 3, for a given Legendre immersion, we introduce a suitable linear connection on the trivial bundle $M^2 \times \mathbb{R}^3$ and show that the existence of non-zero parallel sections with respect to this connection is a necessary and sufficient condition for the Legendre immersion to have non-trivial deformations. In particular, we show that the non-trivial deformations of nondegenerate Legendre immersion depend on three parameters at most. The above characterization allows the introduction of free parameters in the Maurer-Cartan form of the canonical frame of a deformable surface without violating the structure equations, and suggests the existence of a Bäcklund transformation for the class of deformable Legendre surfaces. We will return on this topic elsewhere.

In Section 4, we discuss some examples of deformable Legendre immersions.

In the last section, we study the existence of deformations. We use the characterization of deformable Legendre immersions in terms of parallel sections to set up the exterior differential system of a deformation. We then prove that this system is in involution in Cartan's sense and that its general solutions depend on six functions in one variable.

1. Surface theory in Lie sphere geometry

In this section, we briefly recall the basic structure of Lie sphere geometry and develop the method of moving frames for immersed surfaces in the context of that geometry. More details about Lie sphere geometry are given in the recent monograph of Cecil [8], in the book of Blaschke [4], or in Lie's original work [16].

1.1. **Legendre immersions.** Let S^3 be the unit sphere in $\mathbb{R}^4$ and identify the unit tangent bundle $\Lambda = T_1 S^3$ with the set of all pairs $(v, \xi) \in S^3 \times S^3$ such that v is orthogonal to ξ, i.e.,

$$\Lambda = \{(v, \xi) \in S^3 \times S^3 \subset \mathbb{R}^4 \times \mathbb{R}^4 \,|\, v \cdot \xi = 0\}.$$

Let π_1, $\pi_2 : T_1 S^3 \to S^3$ denote the restrictions to Λ of the canonical projections of $S^3 \times S^3$ onto its factors. Then the equation $d\pi_1 \cdot \pi_2 = 0$ defines a 4-dimensional contact distribution $\mathcal{D}$ on Λ.

If $F : M^2 \to S^3$ is an immersed surface oriented by a field of unit normals n, then $(F, n) : M^2 \to \Lambda$ is an integral submanifold of $\mathcal{D}$. In general, an immersion $f : M^2 \to \Lambda$ which is an integral submanifold of $\mathcal{D}$ is called a *Legendre surface*. The *Lie sphere group*, that is, the group generated by the conformal transformations of S^3 together with the group of normal shifts, which transform an oriented surface to its parallel surface at a fixed oriented distance in the normal direction, preserves the contact distribution $\mathcal{D}$ and acts naturally on the space of Legendre surfaces.

If $f = (F, n) : M^2 \to \Lambda$ is a Legendre immersion, the smooth map $F : M^2 \to S^3$ need not be an immersion. However, without loss of generality, we can always assume (applying if necessary a normal shift) that F is locally an immersion. This follows from a result of Pinkall [23] (see also [8]) stating that, for each $p \in M^2$, there exists $t \in [0, \pi)$ for which the parallel surface $F_t = \cos t \, F + \sin t \, n$ is locally an immersion.

We recall that the *curvature sphere* at $p \in M^2$ corresponding to a principal curvature k_i is the oriented sphere in oriented contact with $F(M^2)$ at $F(p)$ and centered at the focal point determined by the principal curvature k_i. The notion

of a curvature sphere is invariant under Lie transformations. For instance, if $F : M^2 \to \mathbb{R}^3$ is an immersed surface oriented by the field of normals $n : M^2 \to S^2$, then $(F, n) : M^2 \to \mathbb{R}^3 \times S^2 \subset \Lambda$ is a Legendre surface. In this case, the curvature spheres at a point $p \in M^2$ are the oriented spheres $\sigma_i(p)$ centered at $F(p) + k_i(p)^{-1} n(p)$, with signed radius $k_i(p)^{-1}$, $i = 1, 2$. When $k_i(p) = 0$, then σ_i is the oriented tangent plane of the surface at $F(p)$.

Definition 1.1 (Nondegeneracy condition). We say that a Legendre surface $f = (F, n) : M^2 \to \Lambda$ is *nondegenerate* if F is umbilic free and both the curvature spheres corresponding to the principal curvatures k_1, k_2 are immersions into the space of oriented 2-spheres in S^3 (including points).

1.2. **Moving Lie frames for Legendre surfaces.** Let $\mathbb{R}^{4,2}$ denote $\mathbb{R}^6$ with the symmetric bilinear form

(1.1) $\qquad \langle X, Y \rangle = -(x^0 y^5 + x^5 y^0) - (x^1 y^4 + x^4 y^1) + x^2 y^2 + x^3 y^3 = {}^t X g Y$

of signature $(4, 2)$, where (x^J) and (y^J) are the coordinates of X and Y with respect to the standard basis $(\epsilon_0, \ldots, \epsilon_5)$ of $\mathbb{R}^6$. Let G be the connected component of the identity of the group

$$\{A \in \mathrm{GL}(6, \mathbb{R}) \,|\, {}^t A g A = g\}$$

of linear transformations which leave the form (1.1) invariant and let $\mathfrak{g} = \{B \in \mathfrak{gl}(6, \mathbb{R}) \,|\, {}^t B g + g B = 0\}$ be its Lie algebra. For each $A \in G$, we denote by $A_J = A \cdot \epsilon_J$ the J-th column vector of A. Regarding each of the vectors A_J as a vector-valued function $v : G \to \mathbb{R}^6$ on G, since the A_J form a basis, there exist unique 1-forms ω_J^I, $I, J \in \{0, 1, \ldots, 5\}$, so that

(1.2) $\qquad\qquad\qquad dA_J = \omega_J^I A_I, \quad J = 0, \ldots, 5.$

(We use the summation convention on repeated indices.) The 1-forms ω_J^I are the components of the left-invariant Maurer–Cartan form $\omega = A^{-1} dA$ of G. They are connected by relations obtained from the differentiation of $\langle A_I, A_J \rangle = g_{IJ}$, $I, J \in \{0, 1, \ldots, 5\}$, which are

(1.3) $\qquad {}^t \omega g + g \omega = 0, \quad \text{or} \quad \omega_I^K g_{KJ} + \omega_J^K g_{KI}, \quad I, J \in \{0, 1, \ldots, 5\},$

and reflect the structure of the Lie algebra $\mathfrak{g}$. The forms

$$\omega_0^0,\ \omega_1^1,\ \omega_1^0,\ \omega_0^1,\ \omega_0^2,\ \omega_0^3,\ \omega_1^2,\ \omega_1^3,\ \omega_0^4,\ \omega_2^3,\ \omega_2^0,\ \omega_3^0,\ \omega_2^1,\ \omega_3^1,\ \omega_4^0$$

yield a left-invariant coframe field on G. Differentiating (1.2), we obtain the structure equations of G, which are

(1.4) $\qquad d\omega = -\omega \wedge \omega, \quad \text{or} \quad d\omega_J^I = -\omega_K^I \wedge \omega_J^K, \quad I, J \in \{0, 1, \ldots, 5\}.$

For each $X \in G$, the Maurer–Cartan form ω transforms as follows

(1.5) $\qquad\qquad\qquad R_X^*(\omega) = X^{-1} \omega X.$

The projectivization $\mathcal{Q} = \mathbb{P}(\mathcal{L})$ of the light cone $\mathcal{L}$ of $\mathbb{R}^{4,2}$ is known in the classical literature as the *Lie quadric*. In Lie sphere geometry, the Lie quadric parametrizes the set of all oriented 2-sphere in S^3, including points, and the lines in $\mathcal{Q}$ correspond to parabolic pencils of spheres in oriented contact. The set of all lines in $\mathcal{Q}$, that is, the isotropic Grassmannian of null 2-planes through the origin in $\mathbb{R}^{4,2}$, forms a smooth manifold which can be identified with Λ (for more details see [8]). Under this identification, the group G acts transitively on Λ by the usual action of G on

the Grassmannian $G_2(\mathbb{R}^{4,2})$ and preserves the contact structure. The projection map

(1.6) $$\pi_\Lambda : A \in G \to A[\epsilon_0 \wedge \epsilon_1] = [A_0 \wedge A_1] \in \Lambda = G/G_0,$$

defines a principal G_0-bundle over Λ, where G_0 is the isotropy subgroup at the chosen origin $[\epsilon_0 \wedge \epsilon_1]$.

A Legendre surface $f : M^2 \to \Lambda$ may then be represented by two maps $F_0, F_1 : M^2 \to \mathcal{L}$ such that $f = [F_0 \wedge F_1]$, $\langle dF_0, F_1 \rangle = 0$ and $\langle F_0, F_1 \rangle = 0$. Of course, such a representation is not unique. For example, if $F : M^2 \to \mathbb{R}^3$ is any smooth immersion in $\mathbb{R}^3$, oriented by a field of unit normals $n : M^2 \to S^2$, then the *Legendre lift* $f = [F_0 \wedge F_1]$ is given by

(1.7) $$\begin{cases} F_0 = {}^t\left(1, \frac{1}{\sqrt{2}}F^1, F^2, F^3, -\frac{1}{\sqrt{2}}F^1, \frac{1}{2}F \cdot F\right), \\ F_1 = \frac{1}{\sqrt{2}}{}^t\left(0, \frac{1}{2}(1+n^1), n^2, n^3, \frac{1}{\sqrt{2}}(1-n^1), n \cdot F\right). \end{cases}$$

A *frame field* along a Legendre surface $f : M^2 \to \Lambda$ is a smooth map $A : U \to G$ defined on some open subset of M^2 such that $f = [A_0 \wedge A_1]$. For each local frame $A : U \to G$ we let $\alpha = A^*\omega$. The Legendre condition simply means that the form α_0^4 vanishes identically. Any other local frame is given by $\widetilde{A} = A \cdot X$, for some smooth map $X : U \to G_0$, and the 1-forms α and $\widetilde{\alpha}$ are related by

(1.8) $$\widetilde{\alpha} = X^{-1}dX + X^{-1}\alpha X.$$

The totality of frames along f is the principal G_0-bundle $\mathcal{F}_0(f) \to M^2$, where

$$\mathcal{F}_0(f) = \{(p, A) \in M \times G \mid f(p) = [A_0 \wedge A_1]\}.$$

The canonical frame. Following the usual practice in the method of moving frames, we can construct a canonical lift to the group $G/\mathbb{Z}_2$ for any nondegenerate Legendre surface. The idea of the procedure is at each step to normalize the Maurer–Cartan matrix of a frame along f as much as possible, and then take the exterior derivative of the equations expressing this normalization, thereby leading to the next step. Similar preferred frames have been given by Blaschke [4] and Ferapontov [9, 11]. Here, we skip the construction.

Theorem 1.2 (Existence of the canonical frame). *Let $f : M^2 \to \Lambda$ be a nondegenerate Legendre immersion of an oriented surface M^2. Then there exists a unique lift $[A] : M^2 \to G/\mathbb{Z}_2$ satisfying the Pfaffian equations*

(1.9) $$\alpha_0^4 = \alpha_0^2 = \alpha_1^3 = \alpha_2^3 = \alpha_0^1 - \alpha_1^2 = \alpha_1^0 - \alpha_0^3 = \alpha_2^0 = \alpha_3^1 = 0$$

with the independence condition

(1.10) $$\alpha_0^3 \wedge \alpha_1^2 > 0.$$

$[A]$ *is called the* canonical frame field *along f. Let $\alpha^1 = \alpha_0^3$ and $\alpha^2 = \alpha_1^2$. The coframe (α^1, α^2) on M^2 is referred to as the* canonical coframe *of f. The bundle of canonical frames along f we will be denoted by $\mathcal{F}(f) \to M^2$.*

Definition 1.3. The *quadratic form* Φ and the *cubic form* Ψ of the immersion f are defined by

(1.11) $$\Phi = -\alpha^1\alpha^2, \qquad \Psi = -(\alpha^1)^3 + (\alpha^2)^3,$$

respectively. The quotient $\mathcal{B} = \Psi/\Phi$ of the cubic form Ψ to the quadratic form Φ is a well defined map $\mathcal{B} : T(M) \to \mathbb{R} \cup \{\infty\}$ which is a rational function when restricted to the tangent spaces $T_p(M)$. $\mathcal{B}$ is called the *Fubini–Blaschke invariant* of f.

The invariant functions. The only non zero components of the Maurer–Cartan form α of the canonical frame are $\alpha^1 = \alpha_0^3$, $\alpha^2 = \alpha_1^2$, and α_0^0, α_1^1, α_2^1, α_1^1, α_3^0, α_4^0. From the exterior differentiation of (1.9) and the structure equations, it follows that there exist smooth functions q_1, q_2, p_1, p_2, and r_1, r_2 such that

(1.12)
$$\begin{cases} \alpha_0^0 = -2q_1\alpha^1 + q_2\alpha^2, & \alpha_1^1 = -q_1\alpha^1 + 2q_2\alpha^2, \\ \alpha_3^0 = r_1\alpha^1 + p_2\alpha^2, & \alpha_2^1 = p_1\alpha^1 + r_2\alpha^2, \\ \alpha_4^0 = -r_2\alpha^1 + r_1\alpha^2, \end{cases}$$

We shall refer to these functions as the *invariant functions* of f. Using the structure equations, we obtain

(1.13)
$$d\alpha^1 = \alpha_0^0 \wedge \alpha^1, \quad d\alpha^2 = \alpha_1^1 \wedge \alpha^2,$$

(1.14)
$$d\alpha_0^0 = (\alpha^2 - \alpha_3^0) \wedge \alpha^1, \quad d\alpha_1^1 = (\alpha^1 - \alpha_2^1) \wedge \alpha^2,$$

(1.15)
$$d\alpha_2^1 = -\alpha_1^1 \wedge \alpha_2^1, \quad d\alpha_3^0 = -\alpha_0^0 \wedge \alpha_3^0, \quad d\alpha_4^0 = -(\alpha_0^0 + \alpha_1^1) \wedge \alpha_4^0.$$

In terms of the invariant functions, these equations become

(1.16)
$$d\alpha^1 = -q_2\alpha^1 \wedge \alpha^2, \quad d\alpha^2 = -q_1\alpha^1 \wedge \alpha^2$$

(1.17)
$$\begin{cases} -2dq_1 \wedge \alpha^1 + dq_2 \wedge \alpha^2 & = & (p_2 - q_1q_2 - 1)\alpha^1 \wedge \alpha^2, \\ -dq_1 \wedge \alpha^1 + 2dq_2 \wedge \alpha^2 & = & (-p_1 + q_1q_2 + 1)\alpha^1 \wedge \alpha^2, \end{cases}$$

(1.18)
$$\begin{cases} dr_1 \wedge \alpha^1 + dp_2 \wedge \alpha^2 & = & (2q_2r_1 + 3q_1p_2)\alpha^1 \wedge \alpha^2, \\ dp_1 \wedge \alpha^1 + dr_2 \wedge \alpha^2 & = & (2q_1r_2 + 3q_2p_1)\alpha^1 \wedge \alpha^2, \\ -dr_2 \wedge \alpha^1 + dr_1 \wedge \alpha^2 & = & 4(q_1r_1 - q_2r_2)\alpha^1 \wedge \alpha^2. \end{cases}$$

Equations (1.16) and (1.17) tell us that the invariant functions q_1, q_2, p_1, and p_2 are determined by the canonical coframe. Equations (1.18) can be viewed as compatibility conditions arising from the fact that the canonical coframe is obtained from the Legendre immersion. Thus, we may think of (1.16) and (1.17) as the Gauss equations and of (1.18) as the Codazzi-Mainardi equations of the immersion.

Relations with Euclidean geometry. Let $f : M^2 \to \Lambda$ be the Legendre lift of an oriented immersion $F : M^2 \to \mathbb{R}^3$ with Gauss map $n = (n^1, n^2, n^3)$ and suppose f is nondegenerate. Let (u, v) be curvature line coordinates on M^2. Then the canonical coframe takes the form

$$\alpha^1 = \frac{1}{k_1 - k_2} \left(\sqrt{e^{-1}g} \, (k_1)_u \, (k_2)_v^2 \right)^{\frac{1}{3}} dv, \quad \alpha^2 = \frac{1}{k_2 - k_1} \left(\sqrt{eg^{-1}} \, (k_1)_u^2 \, (k_2)_v \right)^{\frac{1}{3}} du,$$

where e and g are the coefficients of the first fundamental form of F with respect to the coordinate system (u, v). Moreover, the quadratic and cubic forms takes the form

$$\Phi = \frac{1}{(k_1 - k_2)^2} (k_1)_u (k_2)_v \, du \, dv$$

$$\Psi = -\frac{(k_1)_u \, (k_2)_v}{(k_1 - k_2)^3 \sqrt{eg}} \left(e \, (k_1)_u \, du^3 + g \, (k_2)_v \, dv^3 \right).$$

Observe that $\alpha^1 = (\beta\gamma^2)^{1/3} dv$ and $\alpha^2 = (\beta^2\gamma)^{1/3} du$, where β and γ are given by

$$\beta = \frac{1}{k_1 - k_2} \sqrt{eg^{-1}} \, (k_1)_u, \quad \gamma = \frac{1}{k_2 - k_1} \sqrt{e^{-1}g} \, (k_2)_v.$$

Using the above structure equations, the invariants q_1, q_2, p_1, p_2 can be expressed in terms of β and γ. For example,

$$q_1 = -\frac{1}{3(\beta^2\gamma)^{2/3}} \left(2\beta_v + \frac{\beta}{\gamma}\gamma_v \right), \quad q_2 = \frac{1}{3(\beta\gamma^2)^{2/3}} \left(\frac{\gamma}{\beta}\beta_u + 2\gamma_u \right).$$

The invariants β and γ correspond to the invariants p and q considered by Ferapontov [9], p. 207, and Blaschke [4] in the construction of the canonical frame. Note that the vanishing of both β and γ is equivalent to the condition that the principal curvatures are constant along the corresponding principal foliations. This property characterizes the *cyclides of Dupin*. If one of the two principal curvatures is constant along the corresponding principal foliation, the surface is the envelope of a one-parameter family of oriented spheres (including planes and point-spheres), and we are in presence of a canal surface.

Remark 1.4. Associated with any nondegenerate Legendre immersion f there is the 3-web formed by the *asymptotic lines* of the quadratic form Φ and by the *cyclidic curves*[1], i.e., the curves along the zero-directions of the cubic form Ψ. In view of (1.11), the curves of this web can be defined in terms of the canonical coframe by the Pfaffian equations

$$(1.19) \qquad\qquad \alpha^1 = 0, \quad \alpha^2 = 0, \quad \alpha^1 - \alpha^2 = 0.$$

The connection form of the 3-web is the 1-form ζ_w, uniquely determined by the equations

$$d\alpha^1 = \zeta^w \wedge \alpha^1, \quad d\alpha^2 = \zeta^w \wedge \alpha^2.$$

Equations (1.12) and (1.13) yield

$$(1.20) \qquad\qquad \zeta_w = -q_1\alpha^1 + q_2\alpha^2.$$

From (1.20) one then computes the curvature of the 3-web, which is

$$(1.21) \qquad\qquad R_w = \frac{1}{3}(p_2 - p_1).$$

The surfaces for which the curvature vanishes identically are called diagonally cyclidic (see [4, 9]).

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[1]The family of curves which are orthogonal to the cyclidic curves with respect to the quadratic form Φ is called *anti-cyclidic system*.

2. Contact, deformation and applicability

Let us recall the general notion of deformation [14, 15].

Definition 2.1. Let G/K be a homogeneous space and let $f, \tilde{f} : N \to G/K$ be two smooth maps. We say that f and $\tilde{f}$ are *k-th order deformations of each other* if there exists a smooth map $B : N \to G$ such that, for each point $p \in N$, $\tilde{f}$ and $B(p)f$ have the same k-th order jets at p, i.e., they have analytic contact of second order at p. The map B is said to be a *k-th order deformation*. When B is constant the deformation is said to be *trivial*. A map $f : N \to G/K$ is said to be *deformable of order k* if it admits a non-trivial k-th order deformation.

First, we will express the condition of analytic contact in the special case of mappings from a 2-dimensional manifold M into Λ. For this we need to introduce some notations.

2.1. **Analytic Contact.** Let (x^1, x^2) be a local coordinate system on an open set U of M. Let E be a vector space and let $S^h(U) \otimes E$ denote the symmetric E-valued k-forms on U. The symmetric tensor product of $s \in S^h(U)$ and $t \in S^k(U)$ will be denoted by $s \cdot t$. An element T of $S^h(U) \otimes E$ has a local expression

$$T = T_{i_1 \dots i_h} dx^{i_1} \dots dx^{i_h},$$

where the coefficients $T_{i_1 \dots i_h}$ are E-valued smooth maps, which are totally symmetric in the indices $i_1, \dots, i_h$. We then define the k-th order derivative of T to be the E-valued symmetric form of order $h + k$ given by

$$\delta^k(T) = \frac{\partial^k T_{i_1 \dots i_h}}{\partial x^{i_{h+1}} \dots \partial x^{i_{h+k}}} dx^{i_1} \dots dx^{i_h} dx^{i_{h+1}} \dots dx^{i_{h+k}}.$$

The definition depends on the choice of the local coordinates. Given a pair $T_0, T_1 \in S^h(U) \otimes E$ of E-valued symmetric h-forms and a 2×2 matrix $\rho = (\rho_b^a) \in S^k(U) \otimes \mathfrak{gl}(2, \mathbb{R})$ of symmetric k-forms we set

$$(T_0, T_1)\rho = (\rho_0^0 T_1 + \rho_0^1 T_1, \rho_1^0 T_0 + \rho_1^1 T_1).$$

We can state the following

Lemma 2.2. *Let $f = [F_0 \wedge F_1] : M^2 \to \Lambda$ and $\tilde{f} = [\tilde{F}_0 \wedge \tilde{F}_1] : M \to \Lambda$ be two smooth maps. Then, f and $\tilde{f}$ agree to second order at $p \in M$, i.e., they have the same second order jets at p, if and only if, for every local coordinate system (x^1, x^2) about p, there exist*

$$\rho_0 \in \mathfrak{gl}(2, \mathbb{R}), \quad \rho_1 \in T^*(M)_p \otimes \mathfrak{gl}(2, \mathbb{R}), \quad \rho_2 \in S^2(M)|_p \otimes \mathfrak{gl}(2, \mathbb{R})$$

such that

$$(\tilde{F}_0|_p, \tilde{F}_1|_p) = (F_0|_p, F_1|_p)\rho_0,$$

(2.1) $\quad (\delta \tilde{F}_0|_p, \delta \tilde{F}_1|_p) = (\delta F_0|_p, \delta F_1|_p)\rho_0 + (F_0|_p, F_1|_p)\rho_1,$

$$(\delta^2 \tilde{F}_0|_p, \delta^2 \tilde{F}_1|_p) = (\delta^2 F_0|_p, \delta^2 F_1|_p)\rho_0 + (\delta F_0|_p, \delta F_1|_p)\rho_1 + (F_0|_p, F_1|_p)\rho_2,$$

Proof. Let (x^1, x^2) be a local coordinate system on an open neighborhood U of p. As G acts transitively on Λ, we may assume that

$$f(p) = \tilde{f}(p) = [\epsilon_0 \wedge \epsilon_1].$$

The map

$$y = (y^1, \dots, y^5) \in \mathbb{R}^5 \mapsto [X_0(y) \wedge X_1(y)] \in \Lambda$$

defined by

$$(2.2) \quad \begin{cases} X_0(y) = {}^t\left(1, 0, y^1, y^2, y^3, \frac{1}{2}[(y^1)^2 + (y^2)^2]\right), \\ X_1(y) = {}^t\left(0, 1, y^4, y^5, \frac{1}{2}[(y^4)^2 + (y^5)^2], y^1 y^4 + y^2 y^5 - y^3\right), \end{cases}$$

is a local coordinate system of Λ centered at $[\epsilon_0 \wedge \epsilon_1]$. Then, there exists an open neighborhood $U' \subset U$ of p and smooth maps $h, \tilde{h} : U' \to \mathbb{R}^5$ such that

$$f|_{U'} = [(X_0 \circ h) \wedge (X_1 \circ h)], \quad \tilde{f}|_{U'} = [(X_0 \circ \tilde{h}) \wedge (X_1 \circ \tilde{h})].$$

Thus, f and $\tilde{f}$ have second order analytic contact at p if and only if the maps $G_a = X_a \circ h$ and $\tilde{G}_a = X_a \circ \tilde{h}$, $a = 0, 1$, satisfy

$$(2.3) \quad G_a(p) = \tilde{G}_a(p), \quad \delta G_a|_p = \delta \tilde{G}_a|_p, \quad \delta^2 G_a|_p = \delta^2 \tilde{G}_a|_p, \quad a = 1, 2.$$

Let us write

$$(2.4) \quad (F_0, F_1) = (G_1, G_2)a, \quad (\tilde{F}_0, \tilde{F}_2) = (\tilde{G}_0, \tilde{G}_2)b.$$

where $a, b : U' \to GL(2, \mathbb{R})$ are smooth maps. Using (2.3) and (2.4), a direct computation shows that this is equivalent to (2.1), where ρ_0, ρ_1 and ρ_2 are given by

$$(2.5) \quad \begin{cases} \rho_0 = a(p)b(p)^{-1}, \\ \rho_1 = (\delta a|_p - \rho_0 \delta b|_p)b(p)^{-1}, \\ \rho_2 = \left(\delta^2 a|_p - \rho_0 \delta^2 b|_p - 2\rho_1 \delta b|_p\right) b(p)^{-1}. \end{cases}$$

$\square$

Remark 2.3. From the proof of this lemma we see that f and $\tilde{f}$ have first order analytic contact at p if and only if there exist $\rho_0 \in \mathfrak{gl}(2, \mathbb{R})$ and $\rho_1 \in T^*(M)_p \otimes \mathfrak{gl}(2, \mathbb{R})$ such that

$$\begin{cases} (\tilde{F}_0|_p, \tilde{F}_1|_p) = (F_0|_p, F_1|_p)\rho_0, \\ (\delta \tilde{F}_0|_p, \delta \tilde{F}_1|_p) = (\delta F_0|_p, \delta F_1|_p)\rho_0 + (F_0|_p, F_1|_p)\rho_1. \end{cases}$$

2.2. Deformation of Legendre surfaces.

Notations 2.4. Given two maps $f, \tilde{f} : M^2 \to \Lambda$, let $\mathcal{F}_0(f)$ and by $\mathcal{F}_0(\tilde{f})$ be the G_0-bundles induced on M by f and $\tilde{f}$, respectively. We let $j : \mathcal{F}_0(f) \to G$ and $\tilde{j} : \mathcal{F}_0(\tilde{f}) \to G$ be the natural maps

$$j : (p, A) \in \mathcal{F}_0(f) \to A \in G, \quad \tilde{j} : (p, \tilde{A}) \in \mathcal{F}_0(\tilde{f}) \to \tilde{A} \in G.$$

The pull back of the Maurer-Cartan form of G by j and $\tilde{j}$ will be denoted by $\omega = (\omega_J^I)$ and by $\tilde{\omega} = (\tilde{\omega}_J^I)$, respectively. If $A : U \to G$ and $\tilde{A} : U \to G$ are local cross sections of $\mathcal{F}_0(f)$ and $\mathcal{F}_0(\tilde{f})$, respectively, then the $\mathfrak{g}$-valued 1-forms $A^{-1}dA$ and $\tilde{A}^{-1}d\tilde{A}$ will be denoted by α and $\tilde{\alpha}$.

2.2.1. *Deformations of order zero.* A deformation of order zero between f and $\tilde{f}$ is a smooth map $B : M \to G$ such that $\tilde{f}(p) = B(p)f(p)$, for every $p \in M$. Thus, B induces a bundle isomorphism $\mathcal{B} : \mathcal{F}_0(f) \to \mathcal{F}_0(\tilde{f})$ defined by the formula

$$\mathcal{B} : (p, A) \in \mathcal{F}_0(f) \to (p, B(p)A) \in \mathcal{F}_0(\tilde{f}), \quad \forall (p, A) \in \mathcal{F}_0(f).$$

Conversely, every bundle isomorphism between $\mathcal{F}_0(f)$ and $\mathcal{F}_0(\tilde{f})$ arises from a deformation of order zero.

2.2.2. *First order deformations.*

Theorem 2.5. *A zero-th order deformation $B : M \to G$ of the two maps $f, \tilde{f} :$ $M \to \Lambda$ is of first order if and only if the bundle isomorphism $\mathcal{B} : \mathcal{F}_0(f) \to \mathcal{F}_0(\tilde{f})$ satisfies*

$$(2.6) \quad \tilde{\omega}_0^2 = \mathcal{B}^*(\omega_0^2), \quad \tilde{\omega}_0^3 = \mathcal{B}^*(\omega_0^3), \quad \tilde{\omega}_0^4 = \mathcal{B}^*(\omega_0^4) \quad \tilde{\omega}_1^2 = \mathcal{B}^*(\omega_1^2) \quad \tilde{\omega}_1^3 = \mathcal{B}^*(\omega_1^3).$$

Proof. We have to show that for every local cross section $A : U \to G$ of $\mathcal{F}_0(f)$, the cross section $\tilde{A} = BA : p \in U \to B(p)A(p) \in G$ of $\mathcal{F}_0(\tilde{f})$ satisfies

$$\tilde{\alpha}_0^2 = \alpha_0^2, \quad \tilde{\alpha}_0^3 = \alpha_0^3, \quad \tilde{\alpha}_0^4 = \alpha_0^4 \quad \tilde{\alpha}_1^2 = \alpha_1^2 \quad \tilde{\alpha}_1^3 = \alpha_1^3.$$

Recall that B is a first order deformation if and only if the two maps $\tilde{f}$ and $B(p)f$ have first order analytic contact at p, for each point $p \in M$. The map $A' = B(p)A :$ $U \to G$ is a frame along $B(p)f$ and B is a first order deformation if and only if the maps

$$F' : q \in U \to [A'_0|_q \wedge A'_1|_q] \in \Lambda, \quad F : q \in U \to [\tilde{A}_0|_q \wedge \tilde{A}_1|_q] \in \Lambda$$

have first order analytic contact at p. From the characterization of analytic contact, it follows that F and F' have first order analytic contact at p if and only if there exist $\rho_0(p) \in \mathfrak{gl}(2, \mathbb{R})$ and $\rho_1(p) \in T^*(M)_p \otimes \mathfrak{gl}(2, \mathbb{R})$ such that

$$(2.7) \qquad \left\{ \begin{array}{l} (\tilde{A}_0|_p, \tilde{A}_1|_p) = (A'_0|_p, A'_1|_p)\rho_0(p), \\ (\delta \tilde{A}_0|_p, \delta \tilde{A}_0|_p) = (\delta A'_0|_p, \delta A'_1|_p)\rho_0(p) + (A'_0|_p, A'_1|_p)\rho_1(p). \end{array} \right.$$

Since A' and $\tilde{A}$ agree at p, we then have

$$(2.8) \qquad\qquad\qquad\qquad \rho_0 = \mathrm{Id}_{2 \times 2}.$$

Now, the structure equations of G imply

$$(2.9) \qquad dA'_0 = \alpha_0^J A'_J, \ dA'_1 = \alpha_1^J A'_J, \ d\tilde{A}_0 = \tilde{\alpha}_0^J \tilde{A}_J, \ d\tilde{A}_1 = \tilde{\alpha}_1^J \tilde{A}_J, \ J = 0, \dots, 5.$$

Substituting (2.9) into (2.7) yields

$$\tilde{\alpha}_0^2|_p = \alpha_0^2|_p, \quad \tilde{\alpha}_0^3|_p = \alpha_0^3|_p, \quad \tilde{\alpha}_0^4|_p = \alpha_0^4|_p, \quad \tilde{\alpha}_1^2|_p = \alpha_1^2|_p, \quad \tilde{\alpha}_1^3|_p = \alpha_1^3|_p$$

and

$$\rho_1(p) = \left(\begin{array}{cc} \tilde{\alpha}_0^0|_p - \alpha_0^0|_p & \tilde{\alpha}_1^0|_p - \alpha_1^0|_p \\ \tilde{\alpha}_0^1|_p - \alpha_0^1|_p & \tilde{\alpha}_1^1|_p - \alpha_1^1|_p \end{array} \right).$$

Since p has been chosen arbitrarily, we can conclude that the equations

$$\tilde{\alpha}_0^2 = \alpha_0^2, \quad \tilde{\alpha}_0^3 = \alpha_0^3, \quad \tilde{\alpha}_0^4 = \alpha_0^4, \quad \tilde{\alpha}_1^2 = \alpha_1^2, \quad \tilde{\alpha}_1^3 = \alpha_1^3$$

are identically satisfied on U. This gives the required result. $\qquad\qquad \square$

As an application of Theorem 2.5 we have

Corollary 2.6. *If $f, \tilde{f} : M^2 \to \Lambda$ are first order deformations of each other then f is a Legendre immersion if and only if $\tilde{f}$ is a Legendre immersion.*

2.2.3. *Second order deformations.* We begin by proving the following

Theorem 2.7. *Let $f, \tilde{f} : M \to \Lambda$ be two nondegenerate Legendre immersions. Then f and $\tilde{f}$ are second order deformations of each other if and only if there exists a bundle isomorphism*

$$\mathcal{B} : \mathcal{F}(f) \to \mathcal{F}(\tilde{f})$$

such that

$$(2.10) \qquad \tilde{\omega}_0^3 = \mathcal{B}^*(\omega_0^3), \quad \tilde{\omega}_1^2 = \mathcal{B}^*(\omega_1^2).$$

Proof. Let $U \subset M$ be any coordinate neighborhood of M and let $A : U \to G$ be a canonical frame along f. We show that a first order deformation $B : M^2 \to G$ is of the second order if and only if $\tilde{A} = BA : U \to G$ is a canonical frame along $\tilde{f}$ such that

$$\tilde{\alpha}_0^3 = \alpha_0^3, \quad \tilde{\alpha}_1^2 = \alpha_1^2.$$

From Theorem 2.5, we know that $\tilde{A}$ is such that

$$(2.11) \qquad \tilde{\alpha}_0^3 = \alpha_0^3, \quad \tilde{\alpha}_1^2 = \alpha_1^2.$$

We also know that the frame fields $\tilde{A}$ and $A' = B(p)A$ satisfy

$$(2.12) \qquad \begin{cases} (\tilde{A}_0, \tilde{A}_1)|_p = (A_0', A_1'), \\ (\delta\tilde{A}_0, \delta\tilde{A}_0)|p = (\delta A_0', \delta A_1')|_p + (A_0', A_1')|_p \rho_1 \end{cases}$$

where

$$\rho_1 = \begin{pmatrix} (\tilde{\alpha}_0^0 - \alpha_0^0)|_p & (\tilde{\alpha}_1^0 - \alpha_1^0)|_p \\ (\tilde{\alpha}_0^1 - \alpha_0^1)|_p & (\tilde{\alpha}_1^1 - \alpha_1^1)|_p \end{pmatrix}.$$

Lemma 2.2 implies that B is a second order deformation if and only if, for every $p \in U$, there exists

$$\rho_2 = \begin{pmatrix} \sigma_0^0 & \sigma_1^0 \\ \sigma_0^1 & \sigma_1^1 \end{pmatrix} \in S^2(M)|_p \otimes \mathfrak{gl}(2, \mathbb{R})$$

such that

$$(2.13) \qquad (\delta^2 \tilde{A}_0, \delta^2 \tilde{A}_1)|_p = (\delta^2 A_0', \delta^2 A_1')|_p + 2(\delta A_0', \delta A_1')|_p \rho_1 + (A_0', A_1')|_p \rho_2.$$

Equation (2.13), when written out, gives

$$(2.14) \qquad \begin{cases} \delta^2 \tilde{A}_0|_p = \delta^2 A_0'|_p + 2\delta A_0'|_p (\tilde{\alpha}_0^0 - \alpha_0^0)|_p + \delta A_1'|_p (\tilde{\alpha}_0^1 - \alpha_0^1)|_p \\ \qquad\quad + A_0'|_p \sigma_0^0 + A_1'|_p \sigma_0^1, \\ \delta^2 \tilde{A}_1|_p = \delta^2 A_1'|_p + 2\delta A_0'|_p (\tilde{\alpha}_1^0 - \alpha_1^0)|_p + \delta A_1'|_p (\tilde{\alpha}_1^1 - \alpha_1^1)|_p \\ \qquad\quad + A_0'|_p \sigma_1^0 + A_1'|_p \sigma_1^1. \end{cases}$$

On the other hand, from the Maurer-Cartan equations of $\tilde{A}$ and the fact that $\tilde{A}|_p = A'|_p$, we compute

$$(2.15) \qquad \begin{cases} \delta^2 \tilde{A}_0|_p = \gamma_0^0|_p A_0'|_p + \gamma_0^1|_p A_1'|_p + \gamma_0^2|_p A_2'|_p + \gamma_0^3|_p A_3'|_p + \gamma_0^5|_p A_5'|_p, \\ \delta^2 \tilde{A}_1|_p = \gamma_1^0|_p A_0'|_p + \gamma_1^1|_p A_1'|_p + \gamma_1^2|_p A_2'|_p + \gamma_1^3|_p A_3'|_p + \gamma_1^4|_p A_4'|_p, \end{cases}$$

where

$$\begin{cases} \gamma_0^0 &=& \delta\tilde\alpha_0^0 + \tilde\alpha_0^0\tilde\alpha_0^0 + \tilde\alpha_0^1\tilde\alpha_1^0 + \tilde\alpha_0^3\tilde\alpha_3^0, \\ \gamma_0^1 &=& \delta\tilde\alpha_0^1 + \tilde\alpha_0^0\tilde\alpha_0^1 + \tilde\alpha_0^1\tilde\alpha_1^0 + \tilde\alpha_0^3\tilde\alpha_3^1, \\ \gamma_0^2 &=& \tilde\alpha_0^1\tilde\alpha_1^2 + \tilde\alpha_0^3\tilde\alpha_3^2, \\ \gamma_0^3 &=& \delta\tilde\alpha_0^3 + \tilde\alpha_0^0\tilde\alpha_0^3, \\ \gamma_0^5 &=& \tilde\alpha_0^3\tilde\alpha_0^3, \\ \gamma_1^0 &=& \delta\tilde\alpha_1^0 + \tilde\alpha_1^0\tilde\alpha_0^0 + \tilde\alpha_1^1\tilde\alpha_1^0 + \tilde\alpha_1^2\tilde\alpha_2^0, \\ \gamma_1^1 &=& \delta\tilde\alpha_1^1 + \tilde\alpha_1^0\tilde\alpha_0^1 + \tilde\alpha_1^1\tilde\alpha_1^1 + \tilde\alpha_1^2\tilde\alpha_2^1, \\ \gamma_1^2 &=& \delta\tilde\alpha_1^2 + \tilde\alpha_1^1\tilde\alpha_1^2, \\ \gamma_1^3 &=& \tilde\alpha_1^0\tilde\alpha_0^3 + \tilde\alpha_1^2\tilde\alpha_2^3, \\ \gamma_1^4 &=& \tilde\alpha_1^2\tilde\alpha_1^2. \end{cases}$$

Using the Maurer-Cartan equations of A', (2.14) can be written as

$$(2.16) \qquad \begin{cases} \delta^2\tilde A_0'|_p = \beta_0^0|_p A_0'|_p + \beta_0^1|_p A_1'|_p + \beta_0^2|_p A_2'|_p + \beta_0^3|_p A_3'|_p + \gamma_0^5|_p A_5'|_p, \\ \delta^2\tilde A_1'|_p = \beta_1^0|_p A_0'|_p + \beta_1^1|_p A_1'|_p + \beta_1^2|_p A_2'|_p + \beta_1^3|_p A_3'|_p + \beta_1^4|_p A_4'|_p, \end{cases}$$

where

$$\begin{cases} \beta_0^0 &=& (\delta\alpha_0^0 + \alpha_0^0\alpha_0^0 + \alpha_0^1\alpha_1^0 + \alpha_0^3\alpha_3^0 + 2\alpha_0^0(\tilde\alpha_0^0 - \alpha_0^0) + 2\alpha_1^0(\tilde\alpha_0^1 - \alpha_0^1) + \sigma_0^0)|_p, \\ \beta_0^1 &=& (\delta\alpha_0^1 + \alpha_0^0\alpha_0^1 + \alpha_0^3\alpha_3^1 + 2\alpha_0^0(\tilde\alpha_0^0 - \alpha_0^0) + 2\alpha_1^1(\tilde\alpha_0^1 - \alpha_0^1) + \sigma_0^1)|_p, \\ \beta_0^2 &=& (\alpha_0^1\alpha_1^2 + 2\alpha_1^2(\tilde\alpha_0^1 - \alpha_0^1))|_p, \\ \beta_0^3 &=& (\delta\alpha_0^3 + \alpha_0^0\alpha_0^3 + 2\alpha_0^3(\tilde\alpha_0^0 - \alpha_0^0))|_p, \\ \beta_0^5 &=& (\alpha_0^3\alpha_0^3)|_p, \\ \beta_1^0 &=& (\delta\alpha_1^0 + \alpha_1^0\alpha_0^0 + \alpha_1^1\alpha_1^0 + \alpha_1^2\alpha_2^0 + 2\alpha_0^0(\tilde\alpha_1^0 - \alpha_1^0) + 2\alpha_1^0(\tilde\alpha_1^1 - \alpha_1^1) + \sigma_1^0)|_p, \\ \beta_1^1 &=& (\delta\alpha_1^1 + \alpha_1^0\alpha_0^1 + \alpha_1^1\alpha_1^1 + \alpha_1^2\alpha_2^1 + 2\alpha_0^1(\tilde\alpha_1^0 - \alpha_1^0) + 2\alpha_1^1(\tilde\alpha_1^1 - \alpha_1^1) + \sigma_1^1)|_p, \\ \beta_1^2 &=& (\delta\alpha_1^2 + \alpha_1^1\alpha_1^2 + 2\alpha_1^2(\tilde\alpha_1^1 - \alpha_1^1))|_p, \\ \beta_1^3 &=& (\alpha_1^0\alpha_0^3 + 2\alpha_0^3(\tilde\alpha_1^0 - \alpha_1^0))|_p, \\ \beta_1^4 &=& (\alpha_1^2\alpha_1^2)|_p. \end{cases}$$

Form (2.15), (2.16) and (2.11), it follows that B is a second order deformation if and only if

$$(2.17) \qquad \begin{cases} \alpha^2(\tilde\alpha_0^1 - \alpha_0^1) &=& -\alpha^1\tilde\alpha_2^3, \\ \alpha^1(\tilde\alpha_1^0 - \alpha_1^0) &=& \alpha^2\tilde\alpha_2^3, \\ \alpha^2(\tilde\alpha_1^1 - \alpha_1^1) &=& 0, \\ \alpha^1(\tilde\alpha_0^0 - \alpha_0^0) &=& 0. \end{cases}$$

The last two equations of (2.17) yield

$$(2.18) \qquad\qquad \tilde\alpha_0^0 = \alpha_0^0, \quad \tilde\alpha_1^1 = \alpha_1^1.$$

Computing the exterior derivatives of (2.18) and using the structure equations, we obtain

$$(2.19) \qquad\qquad \tilde\alpha_0^1 \wedge \alpha_1^2 = \tilde\alpha_1^0 \wedge \alpha_0^3 = 0.$$

Differentiating the equations in (2.17), and using again the structure equations, we see that

$$\begin{cases} \alpha^2 \wedge (\tilde\alpha_0^1 - \alpha_0^1) + \alpha^1 \wedge \tilde\alpha_2^3 = 0, \\ \alpha^1 \wedge (\tilde\alpha_1^0 - \alpha_1^0) - \alpha^2 \wedge \tilde\alpha_2^3 = 0. \end{cases}$$

This implies that $\tilde\alpha_2^3$ vanishes identically, Furthermore, from the first two equations of (2.17) we get

$$\tilde\alpha_0^1 - \alpha_0^1 = \tilde\alpha_1^0 - \alpha_1^0 = 0.$$

This yields the required result.

$$\square$$

Taking into account (1.12), we have

Corollary 2.8. *Let $f, \tilde{f}$ be nondegenerate Legendre surfaces and let A be a canonical frame along f. Then $f, \tilde{f}$ are non-trivial second order deformations of each other if and only if there exists a normal frame $\widetilde{A}$ along $\tilde{f}$ such that*

$$(2.20) \qquad \alpha^1 = \tilde{\alpha}^1, \quad \alpha^2 = \tilde{\alpha}^2,$$

$$(2.21) \qquad \alpha_3^0 - \tilde{\alpha}_3^0 = w_1 \alpha^1, \quad \alpha_2^1 - \tilde{\alpha}_2^1 = w_2 \alpha^2, \quad \alpha_4^0 - \tilde{\alpha}_4^0 = -w_2 \alpha^1 + w_1 \alpha^2,$$

for smooth functions w_1, w_2 such that $(w_1)^2 + (w_2)^2 \neq 0$.

Remark 2.9. If $f, \tilde{f}$ are non-trivial second order deformations of each other, then from the structure equations of the canonical frames A and $\widetilde{A}$, it follows that the $\mathfrak{g}$-valued 1-form $\eta = \tilde{\alpha} - \alpha$ satisfies

$$(2.22) \qquad \boxed{d\eta + \alpha \wedge \eta + \eta \wedge \alpha = 0.}$$

We can summarize the previous results in the following.

Theorem 2.10. *Let $f, \tilde{f} : M^2 \to \Lambda$ be non-congruent, nondegenerate Legendre immersions. Then, the following statements are equivalent:*

(1) *f and $\tilde{f}$ are non-trivial second order deformations of each other.*
(2) *there exists a bundle isomorphism $\mathcal{B} : \mathcal{F}(f) \to \mathcal{F}(\tilde{f})$ such that*

$$\omega^1 = \mathcal{B}^*(\tilde{\omega}^1), \quad \omega^2 = \mathcal{B}^*(\tilde{\omega}^2).$$

(3) *f and $\tilde{f}$ induce the same canonical coframe on M.*
(4) *f and $\tilde{f}$ have the same quotient of cubic to quadratic forms, that is,*

$$(2.23) \qquad \Psi/\Phi = \widetilde{\Psi}/\widetilde{\Phi}.$$

The equivalence of (1), (2), and (3) is a direct consequence of Theorem 2.7. As for the equivalence with (4), if (2.23) holds, there exist canonical frames A and $\widetilde{A}$ along f and $\tilde{f}$, respectively, such that $\tilde{\alpha}^1 = \alpha^1$, $\tilde{\alpha}^2 = \alpha^2$. Since $f, \tilde{f}$ are not congruent, this yields (2.21).

Example 2.11 (Isothermic nets). Let $U \subset \mathbb{R}^2$ be a simply connected open set with coordinates (u, v). A *net* is a two-parameter smooth immersion $F : U \to \mathbb{R}^3$ satisfying $F_{uv} \cdot F_u \times F_v = 0$. A net F is *isothermic* if F is a conformal map, that is, (u, v) are both principal and isothermal coordinates. Isothermic nets parametrize isothermic surfaces. Examples of isothermic surfaces include quadrics, surfaces of revolution, cones, cylinders and constant mean curvature surfaces. The main local differential invariant of an isothermic net is the *Calapso potential*, that is the positive function φ defined by

$$\varphi^2(du^2 + dv^2) = \frac{1}{4}(k_1 - k_2)^2 dF \cdot dF,$$

where k_1 and k_2 are the principal curvatures. The Gauss-Codazzi equations imply that φ is a solution of the *Calapso-Rothe* equation:

$$\Delta(\varphi^{-1}\varphi_{uv}) + 2(\varphi^2)uv = 0.$$

Denote by $f : U \to \Lambda$ the Legendre lift of F and let $\psi = \log(\varphi)$. If $\psi_u \psi_v \neq 0$, then f is nondegenerate and the corresponding canonical coframe takes the form

$$\alpha^1 = \sqrt[3]{\psi_u(\psi_v)^2}\, dv, \quad \alpha^2 = \sqrt[3]{(\psi_u)^2\psi_v}\, du.$$

Let W be a smooth function such that $\alpha_\varphi = d\left(e^{2\psi}W\right)$, where α_φ is the closed 1-form

$$\alpha_\varphi = -e^{2\psi}\left(\frac{1}{2}\left(e^{-2\psi}\Delta\psi\right)_u + 2\psi_u\left(1 + e^{-2\psi}\Delta\psi\right)\right)du$$

$$+ e^{2\psi}\left(\frac{1}{2}\left(e^{-2\psi}\Delta\psi\right)_v + 2\psi_v\left(1 + e^{-2\psi}\Delta\psi\right)\right)dv.$$

The Calapso potential and the function W give a complete set of invariants for the isothermic net with respect to the conformal group [2, 22], that is, φ and W determine F up to a conformal transformation. One of the most important features of isothermic nets is the existence of a *spectral transformation*. This transformation was independently discovered by P. Calapso and L. Bianchi, who introduced it as the T-*transformation*. Given a real constant $m \in \mathbb{R}$, the T_m-transform $T_m(F)$ of F can be characterized, up to conformal transformations, by

$$\varphi_{T_m(F)} = \varphi_F, \quad W_{T_m(F)} = W_F + me^{-2\psi}.$$

Thus the Legendre lifts of the T-transforms of F have the same canonical coframe and are not congruent (see Remark 4.2 and (4.1)). This shows that the Legendre lifts of isothermic nets are deformable.

Example 2.12 (L-Isothermic nets). Another class of deformable surfaces is given by the Legendre lifts of L-isothermic nets. A net $F : U \to \mathbb{R}^3$ is said to be L-*isothermic* if the Gauss map $n : U \to S^2$ is conformal with respect to the third fundamental form, that is, (u, v) are principal coordinates which are isothermal with respect to the third fundamental form. L-isothermic nets parametrize the class of *L-isothermic surfaces*. Examples include minimal surfaces in $\mathbb{R}^3$ and molding surfaces [20]. The study of L-isothermic surfaces goes back to the work of Blaschke and presents many analogies with that of isothermic surfaces. For instance, L-isothermic surfaces admit a spectral transformation which is the analogue of the T-transformation for isothermic surfaces [18]. We briefly recall some basic properties of L-isothermic nets and show that their Legendre lifts are deformable. The *Blaschke potential* of F is the positive function φ defined by

$$\varphi^2(du^2 + dv^2) = \frac{1}{4}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)^2 dn \cdot dn.$$

In this case the compatibility condition arising from the Gauss-Codazzi equations is the *Blaschke equation*

(2.24)
$$\Delta\left(\frac{1}{\varphi}\varphi_{uv}\right) = 0.$$

Let f be the Legendre lift of F and $\psi = \log(\varphi)$. In the nondegenerate case, which amounts to $\psi_u\psi_v \neq 0$, the canonical coframe of f can be written as

$$\alpha^1 = \sqrt[3]{(\psi_u)^2\psi_v}\,du, \quad \alpha^2 = \sqrt[3]{\psi_u(\psi_v)^2}\,dv.$$

In addition to the Blaschke potential, the other local differential invariant of F is a smooth function W defined by $\alpha_\varphi = d\left(e^{2\psi}W\right)$, where

$$\alpha_\varphi = -e^{2\psi}\left(\frac{1}{2}\left(e^{-2\psi}\Delta\psi\right)_u + 2\psi_u\left(e^{-2\psi}\Delta\psi\right)\right)du$$

$$+ e^{2\psi}\left(\frac{1}{2}\left(e^{-2\psi}\Delta\psi\right)_v + 2\psi_v\left(e^{-2\psi}\Delta\psi\right)\right)dv.$$

Given $m \in \mathbb{R}$, the T_m-transform $T_m(F)$ of F can be characterized (up to Laguerre contact transformations) by

$$\varphi_{T_m(F)} = \varphi_F, \quad W_{T_m(F)} = W_F + me^{-2\psi}.$$

The Legendre lifts of the T-transforms of F have then the same canonical coframe and are not congruent (see Remark 4.2 and (4.1)), from which follows that the Legendre lift of an L-isothermic net is deformable.

3. INFINITESIMAL DEFORMATIONS AND DEFORMABLE SURFACES

Infinitesimal deformations. On the one hand, if f and $\widetilde{f}$ are non-trivial deformations of each other, then $\eta = \widetilde{\alpha} - \alpha$ never vanishes and according to (2.22)

$$(3.1) \qquad\qquad\qquad \delta|_U := A\eta A^{-1}$$

is a closed 1-form with values in $\mathfrak{g}$, for every canonical frame A along f. Moreover,

$$(3.2) \qquad\qquad\qquad D|_U := [\widetilde{A}A^{-1}]$$

defines a smooth map $D : M^2 \to G/\mathbb{Z}_2$ such that

$$D^{-1}dD = \delta.$$

On the other hand, let $f : M^2 \to \Lambda$ be a nondegenerate Legendre immersion and define $\eta(w_1, w_2) \in \Omega^1(U) \otimes \mathfrak{g}$ by

$$(3.3) \quad \eta(w_1, w_2) = \begin{pmatrix} 0 & 0 & 0 & w_1\alpha^1 & -w_2\alpha^1 + w_1\alpha^2 & 0 \\ 0 & 0 & w_2\alpha^2 & 0 & 0 & w_2\alpha^1 - w_1\alpha^2 \\ 0 & 0 & 0 & 0 & w_2\alpha^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_1\alpha^1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for smooth functions w_1, w_2. Note that η takes values in the abelian subalgebra

$$\mathfrak{a} = \{T \in \mathfrak{g} \mid T(\epsilon_0) = T(\epsilon_1) = 0, \, T(\epsilon_2) \propto \epsilon_1, \, T(\epsilon_3) \propto \epsilon_0\}.$$

From (2.22), it follows that the 1-form $A\eta A^{-1} \in \Omega^1(U) \otimes \mathfrak{g}$ is independent of A. Thus, there exists $\delta \in \Omega^1(M) \otimes \mathfrak{g}$ such that $\delta|_U = A\eta A^{-1}$.

Definition 3.1 (Infinitesimal deformations)**.** We say that η is an *infinitesimal deformation* of f if δ is closed. Let Δ_f denote the set of infinitesimal deformations of f.

We are now in a position to characterize the Legendre surfaces which admit non-trivial deformations in terms of infinitesimal deformations.

Theorem 3.2. *Let M^2 be simply connected. Then a nondegenerate Legendre immersion $f : M^2 \to \Lambda$ admits non-trivial deformations if and only if $\Delta_f \neq 0$.*

Proof. If $\widetilde{f}$ is a non-trivial deformation of f, then $\eta = \widetilde{\alpha} - \alpha$ defines a non-zero infinitesimal deformation. Conversely, let η be a non zero infinitesimal deformation. Then, $\delta \in \Omega^1(M) \otimes \mathfrak{g}$ is a non-zero closed 1-form and there exists $D : M \to G$ such that $D^{-1}dD = \delta$. Note that the map $\widetilde{f} : p \in M \mapsto D(p) \cdot f(p) \in \Lambda$ is a non-trivial deformation of f. $\qquad\square$

Remark 3.3. Note that for every non-zero $\eta \in \Delta_f$ there exists a non-trivial deformation $\widetilde{f}$ which is uniquely defined by f and the corresponding infinitesimal deformation, up to the action of G. Moreover, as Δ_f is a real vector space, given $\eta \in \Delta_f$ and $r \in \mathbb{R}$, $r\eta$ is another infinitesimal deformation. Therefore, the deformations of f arise in one-parameter families. In other words, deformable surfaces do have a *spectral transformation.* This suggests the existence of a Bäcklund transformation for deformable surfaces.

Infinitesimal deformations and parallel sections. Let $f : M^2 \to \Lambda$ be a nondegenerate Legendre immersion and consider the $\mathfrak{gl}(3, \mathbb{R})$-valued 1-form

$$(3.4) \qquad \sigma = \begin{pmatrix} -2(2q_1\alpha^1 - q_2\alpha^2) & 0 & -\alpha^1 \\ 0 & -2(q_1\alpha^1 - 2q_2\alpha^2) & \alpha^2 \\ 2(p_2 - 1)\alpha^2 & -2(p_1 - 1)\alpha^1 & -3(q_1\alpha^1 - q_2\alpha^2) \end{pmatrix}.$$

Definition 3.4 (σ-connection)**.** The form (3.4) defines a linear connection

$$D^\sigma w := dw + \sigma w$$

on the trivial bundle $M^2 \times \mathbb{R}^3$, for each smooth function $w : M^2 \to \mathbb{R}^3$. D^σ is referred to as the *σ-connection* of f. By $\mathcal{P}_f$ we denote the vector space consisting of all *parallel sections* of the σ-connection.

A simple computation shows that the curvature form Ω^σ of the σ-connection is given by

$$(3.5) \qquad\qquad \Omega^\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\partial_1 p_2 & 2\partial_2 p_1 & 3(p_2 - p_1) \end{pmatrix} \alpha^1 \wedge \alpha^2,$$

where for a smooth function $g : M \to \mathbb{R}$ we write $dg = \partial_1 g\,\alpha^1 + \partial_2 g\,\alpha^2$.

For every $w = (w_1, w_2, w_3) : M^2 \to \mathbb{R}^3$, let $\eta(w_1, w_2)$ be defined by (3.3).

Proposition 3.5. *A nondegenerate Legendre immersion f is deformable if and only if there exists a parallel section of the σ-connection. Moreover, the mapping*

$$w \in \mathcal{P}_f \to \eta(w_1, w_2) \in \Delta_f$$

is an isomorphism of vector spaces.

Proof. Let $\widetilde{f}$ be a deformation of f. Then $\alpha^1 = \widetilde{\alpha}^1$, $\alpha^2 = \widetilde{\alpha}^2$ and

$$\alpha_3^0 - \widetilde{\alpha}_3^0 = w_1\alpha^1, \quad \alpha_2^1 - \widetilde{\alpha}_2^1 = w_2\alpha^2, \quad \alpha_4^0 - \widetilde{\alpha}_4^0 = -w_2\alpha^1 + w_1\alpha^2,$$

for smooth functions w_1, w_2 such that $(w_1)^2 + (w_2)^2 \neq 0$. Differentiating and using the structure equations, we get

$$d(w_1\alpha^1) = -\alpha_0^0 \wedge w_1\alpha^1,$$
$$d(w_2\alpha^2) = -w_2\alpha_1^1 \wedge \alpha^2,$$
$$d(-w_2\alpha^1 + w_1\alpha^2) = -(\alpha_0^0 + \alpha_1^1) \wedge (-w_1\alpha^1 + w_1\alpha^2),$$

which implies

$$(dw_1 + 2w_1\alpha_0^0) \wedge \alpha^1 = 0,$$
$$(dw_2 + 2w_2\alpha_1^1) \wedge \alpha^2 = 0,$$
$$(dw_1 + 2w_1\alpha_0^0) \wedge \alpha^2 - (dw_2 + 2w_2\alpha_1^1) \wedge \alpha^1 = 0,$$

By Cartan's Lemma, there exists a smooth function $w_3 : M^2 \to \mathbb{R}$ such that

$$dw_1 = -2w_1\alpha_0^0 + w_3\alpha^1, \quad dw_2 = -2w_2\alpha_1^1 - w_3\alpha^2,$$

that is,

$$dw_1 - 2w_1(2q_1\alpha^1 - q_2\alpha^2) - w_3\alpha^1 = 0,$$
$$dw_2 - 2w_2(q_1\alpha^1 - 2q_2\alpha^2) + w_3\alpha^2 = 0.$$

Taking the exterior derivative of these equations yields

$$dw_3 + 2(p_2 - 1)\alpha^2 w_1 - 2(p_1 - 1)\alpha^1 w_2 - 3(q_1\alpha^1 - q_2\alpha^2)w_3 = 0,$$

which shows that (w_1, w_2, w_3) is a parallel cross section of the σ-connection.

The converse follows by observing that $\eta(w_1, w_2)$ defines an infinitesimal deformation if $w = (w_1, w_2, w_3)$ is a parallel section. $\qquad \square$

Remark 3.6. Nondegenerate deformable Legendre surfaces may be classified in terms of the dimension of Δ_f. For a generic f the space Δ_f is one-dimensional. It is not too difficult to show that surfaces with a three-parameter family of deformations can be generically obtained as deformations of the Legendre lifts of molding surfaces in $\mathbb{R}^3$. It is not at all clear if there exist Legendre surfaces with a two-dimensional family of deformations. But, if they exist then they can be reconstructed from the integral manifolds of a Pfaffian system with empty complex characteristic variety. Thus, this class is either empty, or it depends on a finite number of parameters.

4. EXAMPLES

Example 4.1 (Generic deformations)**.** Let f, $\widetilde{f}$ be deformations of each other and let η be the corresponding infinitesimal deformation. According to the notations of Corollary 2.8, we say that the deformation is *generic* if $w_1 w_2$ is nowhere vanishing. In this case, there exist local parameters (u, v) on M such that the canonical coframe takes the form

$$\alpha^1 = \sqrt[3]{(\psi_u)^2 \psi_v}\, du, \quad \alpha^2 = \sqrt[3]{\psi_u(\psi_v)^2}\, dv,$$

where $\psi : M^2 \to \mathbb{R}$ is a smooth function. The T-transforms of isothermic and L-isothermic nets are examples of generic deformations.

Remark 4.2. In the case of an isothermic, respectively, L-isothermic surface, e^ψ is the Calapso, respectively, the Blaschke potential. This can be seen by applying the reduction procedure recalled in Section 1 to their respective conformal and Laguerre canonical frames. For this we need to assume the nondegeneracy condition $\psi_u \neq 0$, $\psi_v \neq 0$.

A direct computation shows that the σ-connection $\sigma = (\sigma_a^b)$ is given by

$$\sigma_2^1 = \sigma_1^2 = 0, \quad \sigma_3^1 = -\alpha^1 = -\sqrt[3]{(\psi_u)^2\psi_v}\,du, \quad \sigma_3^2 = \alpha^2 = \sqrt[3]{\psi_u(\psi_v)^2}\,dv,$$

and

$$\begin{cases} \sigma_1^1 = \frac{4}{3}\left(\frac{\psi_{uu}}{\psi_u} + \frac{2\psi_{uv}}{\psi_v}\right)du + \frac{2}{3}\left(\frac{2\psi_{uv}}{\psi_u} + \frac{\psi_{vv}}{\psi_v}\right)dv \\ \sigma_2^2 = \frac{2}{3}\left(\frac{\psi_{uu}}{\psi_u} + \frac{2\psi_{uv}}{\psi_v}\right)du + \frac{4}{3}\left(\frac{2\psi_{uv}}{\psi_u} + \frac{\psi_{vv}}{\psi_v}\right)dv, \\ \sigma_1^3 = -\frac{2}{\sqrt[3]{(\psi_u)^2\psi_v}}\left(\frac{\psi_{uvv}}{\psi_v} - \frac{\psi_{vv}\psi_{uv}}{(\psi_v)^2}\right)dv, \\ \sigma_2^3 = \frac{2}{\sqrt[3]{\psi_u(\psi_v)^2}}\left(\frac{\psi_{uuv}}{\psi_u} - \frac{\psi_{uu}\psi_{uv}}{(\psi_u)^2}\right)du, \\ \sigma_3^3 = \left(\frac{\psi_{uu}}{\psi_u} + \frac{2\psi_{uv}}{\psi_v}\right)du + \left(\frac{2\psi_{uv}}{\psi_u} + \frac{\psi_{vv}}{\psi_v}\right)dv \end{cases}$$

It is a computational matter to verify that the parallel section $w \in \mathcal{P}_f$ associated to η is given by

$$(4.1) \qquad w = {}^t\left(\sqrt[3]{(\psi_u)^{-4}(\psi_v)^{-2}}, -\sqrt[3]{(\psi_u)^{-2}(\psi_v)^{-4}}, 2(\psi_u)^{-2}(\psi_v)^{-2}\psi_{uv}\right).$$

Remark 4.3. Note that $w^1, w^2 \neq 0$, which characterizes such surfaces. Moreover, in the case of isothermic and L-isothermic nets, w originates the one-parameter family of non-trivial deformations considered in Examples 2.11 and 2.12.

Example 4.4 (Special deformations). A deformation is said to be *special* if $w_1 w_2$ vanishes identically. Deformable surfaces which admit a special deformation play the role of the R_0 surfaces in projective differential geometry. Let $w \in \mathcal{P}_f$ be the parallel section associated to the deformation. Then $w^1 w^2$ and w^3 vanish identically. Two cases may occur: either $w^1 = 0$, or else $w^2 = 0$. Without loss of generality, we assume that $w^2 = 0$. From the structure equations of the canonical frame, it follows that f admits a special deformation with $w^2 = 0$ if and only if $p_2 = 1$. The degree of generality of this class of Legendre immersions will be clear in the last section; we will see that they are rather special. Notice that on M^2 there exist local parameters (u, v) with respect to which the canonical coframe is given by

$$\alpha^1 = \psi^{2/3}du, \quad \alpha^2 = \psi^{1/3}dv,$$

for a smooth function ψ such that $w^1 = \pm\psi^{-2/3}$.

Example 4.5 (Legendre surfaces with 3-parameter families of deformations). In this example we consider the Legendre immersions with flat σ-connection. This example has been discussed by Ferapontov in [11] (see also Finikov [12]). From (3.5) we see that $\Omega^\sigma = 0$ if, and only if, $p_1 = p_2 = c$, for a constant c. According to Remark 1.4, we have

Proposition 4.6. *The 3-web defined by the canonical coframe is flat if and only if* $p_1 = p_2$.

Remark 4.7. From this we infer that deformable diagonally cyclidic surfaces are characterized by having $p_1 = p_2 = \text{const}$.

Since the web-connection is flat, then there exist local coordinates (u, v) such that

(4.2) $$\alpha^1 = e^\psi du, \quad \alpha^2 = e^\psi dv,$$

where ψ is a smooth function. From this we see that

(4.3) $$q_1 = -\psi_u e^{-\psi}, \quad q_2 = \psi_v e^{-\psi}.$$

This implies

(4.4) $$\begin{cases} \alpha_0^0 = 2\psi_u du + \psi_v dv, \\ \alpha_1^1 = \psi_u du + 2\psi_v dv \end{cases}$$

From the structure equation we deduce that ψ is a solution of the Liouville equation

(4.5) $$\psi_{uv} = (1 - c)e^{2\psi}.$$

The other compatibility conditions arising from the structure equations are:

(4.6) $$\begin{cases} d\alpha_2^1 = -\alpha_1^1 \wedge \alpha_2^1, \\ d\alpha_3^0 = -\alpha_0^0 \wedge \alpha_3^0, \\ d\alpha_4^0 = -(\alpha_0^0 + \alpha_1^1) \wedge \alpha_4^0, \end{cases}$$

where

(4.7) $$\begin{cases} \alpha_3^0 = A du + c e^\psi dv, \\ \alpha_2^1 = c e_\psi du + B dv, \\ \alpha_4^0 = -B du + A dv, \end{cases}$$

for two suitable smooth functions A, B (essentially the invariant r_1 and r_2). It is now a computational matter to verify that (4.6) and (4.7) can be written as follows

(4.8) $$\begin{cases} A_v = -A\psi_v + 3c e^\psi \psi_u, \\ B_u = -B\psi_u + 3c e^\psi \psi_v, \\ A_u + B_v = -3A\psi_u - 3B\psi_v. \end{cases}$$

We may rewrite (4.8) in the form

(4.9) $$\begin{cases} dA = (R - 3A\psi_u)du + (3c e^\psi \psi_u - A\psi_u)dv, \\ dB = (3c e^\psi \psi_v - B\psi_u)du - (R + 3B\psi_v)dv, \end{cases}$$

where R is a suitable smooth function. Differentiating (4.9), we get

(4.10) $$\begin{cases} R_u = -R\psi_u - 2(1 - c)e^{2\psi}B - 3c e^\psi(\psi_{vv} + 4\psi_v^2), \\ R_v = -R\psi_v + 2(1 - c)e^{2\psi}A + 3c e^\psi(\psi_{uu} + 4\psi_u^2). \end{cases}$$

The compatibility conditions of this system imply

(4.11) $$3c e^\psi \left(\psi_{uuu} + \psi_{vvv} + 10\psi_u\psi_{uu} + 10\psi_v\psi_{vv} + 8(\psi_u^2 + \psi_v^2) \right) = 0.$$

Two cases may occur: either $c = 0$, or $c \neq 0$. In the first case the only compatibility condition is the Liouville equation, which may be viewed as the soliton equation of this class of surfaces: its solutions depend on two arbitrary functions in one variable. In fact, the general solutions of the Liouville equation are of the form

(4.12) $$\begin{cases} \Phi^2 = \frac{1}{1-c} \frac{\lambda'\dot\mu}{(\lambda+\mu)^2}, & \text{if } c \neq 1, \\ \Phi^2 = \lambda'\dot\mu, & \text{if } c = 1, \end{cases}$$

where $\Phi = e^\psi$, λ is a function of the variable u, and μ is a function of the variable v. Thus, if $c \neq 0$, it follows that ψ is a solution of the overdetermined system

$$(4.13) \qquad \begin{cases} \psi_{uuu} + \psi_{vvv} + 10\psi_u\psi_{uu} + 10\psi_v\psi_{vv} + 8(\psi_u^2 + \psi_v^2) = 0, \\ \psi_{uv} = (1-c)e^{2\psi}. \end{cases}$$

If we use the potential $\Phi = e^\psi$, then (4.13) means that Φ^2 is a function of the form (4.12) such that

$$(4.14) \qquad \left(\frac{1}{\Phi^2} \left(\Phi^2 \left(\Phi^2 \right)_u \right)_u \right)_u + \left(\frac{1}{\Phi^2} \left(\Phi^2 \left(\Phi^2 \right)_v \right)_v \right)_v = 0.$$

Let first examine the case $c = 1$. We take λ and μ as a new variable and we think of λ' and $\dot\mu$ as a functions of λ and μ respectively. With respect to these new coordinates the equation (4.14) is equivalent to

$$(4.15) \qquad \partial^3_{\lambda\lambda\lambda}(\lambda'^3) + \partial^3_{\mu\mu\mu}(\dot\mu^3) = 0.$$

This implies that λ and μ satisfy the ODE

$$(4.16) \qquad \begin{cases} \left(\frac{d\lambda}{du}\right)^3 = P(\lambda), \\ \left(\frac{d\mu}{dv}\right)^3 = Q(\mu), \end{cases}$$

where P and Q are polynomials of order ≤ 3 with the opposite leading coefficients. One may proceed in a similar fashion also in the general case. Substituting

$$(4.17) \qquad \Phi^2 = \frac{1}{1-c} \frac{\lambda'\dot\mu}{(\lambda+\mu)^2}$$

in (4.15), we obtain

$$(4.18) \qquad \begin{aligned} \frac{1}{3}(\lambda+\mu)^3 \left(\frac{d^3\lambda'^3}{d\lambda^3} + \frac{d^3\dot\mu^3}{d\mu^3} \right) &- 4(\lambda+\mu)^2 \left(\frac{d^2\lambda'^3}{d\lambda^2} + \frac{d^2\dot\mu^3}{d\mu^2} \right) \\ &+ 20(\lambda+\mu) \left(\frac{d\lambda'^3}{d\lambda} + \frac{d\dot\mu^3}{d\mu} \right) - 40(\lambda'^3 + \dot\mu^3) = 0. \end{aligned}$$

Applying the operator $\partial^6_{\lambda\lambda\lambda\mu\mu\mu}$ to (4.18), we get

$$(4.19) \qquad \frac{d^6\lambda'^3}{d\lambda^6} + \frac{d^6\dot\mu^3}{d\mu^6} = 0.$$

This implies that λ'^3 and $\dot\mu^3$ are polynomials $P(\lambda)$ and $Q(\mu)$ in λ and μ, respectively, of order ≤ 6. Such polynomials satisfy (4.18) if and only if $Q(T) = -P(-T)$. Thus, λ and μ satisfy the ODE

$$(4.20) \qquad \begin{cases} \left(\frac{d\lambda}{du}\right)^3 = P(\lambda), \\ \left(\frac{d\mu}{dv}\right)^3 = -P(-\mu), \end{cases}$$

where P is a polynomial of degree ≤ 6.

The 3-parameter family of infinitesimal deformations. We finish this example by discussing the 3-parameter family of infinitesimal deformations of such surfaces. The σ-connection is given by

$$(4.21) \qquad \sigma = \begin{pmatrix} 4\psi_u du + 2\psi_v dv & 0 & -e^\psi du \\ 0 & \psi_u du + 4\psi_v dv & e^\psi dv \\ 2(c-1)e^\psi dv & -2(c-1)e^\psi du & 3d\psi \end{pmatrix},$$

where the potential function is given either by

$$(4.22) \qquad e^{2\psi} = \lambda' \dot{\mu},$$

or by

$$(4.23) \qquad e^{2\psi} = \frac{1}{1-c} \frac{\lambda' \dot{\mu}}{(\lambda + \mu)^2}.$$

It is now a computational matter to verify that the parallel sections of the σ-connection are given by:

- If $c = 1$ (i.e., $e^{2\psi} = \lambda' \dot{\mu}$)

$$(4.24) \qquad s_0 e^{-2\psi} \begin{pmatrix} \frac{1}{\lambda'} \\ -\frac{1}{\dot{\mu}} \\ 0 \end{pmatrix} + s_1 e^{-2\psi} \begin{pmatrix} \frac{\lambda}{\lambda'} \\ -\frac{\mu}{\dot{\mu}} \\ e^{-\psi} \end{pmatrix} + s_2 e^{-2\psi} \mu\lambda \begin{pmatrix} \frac{\lambda}{\lambda'} \\ -\frac{\mu}{\dot{\mu}} \\ 2e^{-\psi} \end{pmatrix}$$

where s_0, s_1 and s_3 are real constants.

- If $c \neq 1$ (i.e., $e^{2\psi} = \frac{1}{1-c} \frac{\lambda' \dot{\mu}}{(\lambda + \mu)^2}$), we have

$$(4.25) \qquad s_0 e^{-2\psi} \begin{pmatrix} \frac{-1}{\lambda'} \\ \frac{1}{\dot{\mu}} \\ \frac{2e^{-\psi}}{\lambda + \mu} \end{pmatrix} + s_1 e^{-2\psi} \begin{pmatrix} \frac{\lambda}{\lambda'} \\ \frac{\mu}{\dot{\mu}} \\ \frac{e^{-\psi}(\mu - \lambda)}{\lambda + \mu} \end{pmatrix} + s_2 e^{-2\psi} \begin{pmatrix} \frac{\lambda^2}{\lambda'} \\ \frac{-\mu^2}{\dot{\mu}} \\ \frac{2\mu\lambda e^{-\psi}}{\lambda + \mu} \end{pmatrix},$$

where s_0, s_1 and s_2 are real constants.

5. The differential system of a deformation

It was shown in [15] that the problems of k-th order deformation are equivalent to solving certain exterior differential systems on appropriate spaces. Naturally, for each concrete homogeneous space there is a specific problem to solve. We shall derive this result in the case at hand.

Let $P = (G/\mathbb{Z}_2) \times \mathbb{R}^6 \times \mathbb{R}^3$ and denote by $(q_1, q_2, p_1, p_2, r_1, r_2)$ and (w_1, w_2, w_3) the coordinates on $\mathbb{R}^6$ and $\mathbb{R}^3$, respectively. Let (ω_J^I) be the Maurer-Cartan forms on $G/\mathbb{Z}_2$ and put $\alpha^1 = \omega_0^3$, $\alpha^2 = \omega_1^2$. On P, we consider the exterior differential 1-forms $\eta^1, \ldots, \eta^{16}$ defined by

$$(5.1) \qquad \begin{cases} \eta^1 = \omega_0^4, & \eta^2 = \omega_0^2, & \eta^3 = \omega_1^3, \\ \eta^4 = \omega_2^3, & \eta^5 = \omega_0^1 - \alpha^2, & \eta^6 = \omega_1^0 - \alpha^1, \\ \eta^7 = \omega_2^0, & \eta^8 = \omega_3^1, \end{cases}$$

$$(5.2) \qquad \begin{cases} \eta^9 = \omega_0^0 + 2q_1\alpha^1 - q_2\alpha^2, \\ \eta^{10} = \omega_1^1 + q_1\alpha^1 - 2q_2\alpha^2, \end{cases}$$

$$(5.3) \qquad \begin{cases} \eta^{11} = \omega_3^0 - r_1\alpha^1 - p_2\alpha^2, \\ \eta^{12} = \omega_2^1 - p_1\alpha^1 - r_2\alpha^2, \\ \eta^{13} = \omega_4^0 + r_2\alpha^1 - r_1\alpha^2, \end{cases}$$

$$(5.4) \qquad \begin{cases} \eta^{14} = dw_1 - 2w_1(2q_1\alpha^1 - q_2\alpha^2) - w_3\alpha^1, \\ \eta^{15} = dw_2 - 2w_2(q_1\alpha^1 - 2q_2\alpha^2) + w_3\alpha^2, \\ \eta^{16} = dw_3 + 2w_1(p_2 - 1)\alpha^2 - 2w_2(p_1 - 1)\alpha^1 - 3w_3(q_1\alpha^1 - q_2\alpha^2). \end{cases}$$

Definition 5.1. Let $(\mathcal{I}, \alpha^1 \wedge \alpha^2)$ be the exterior differential system on P generated by $\eta^1, \ldots, \eta^{16}$ with the independence condition $\alpha^1 \wedge \alpha^2 \neq 0$. We call $(\mathcal{I}, \alpha^1 \wedge \alpha^2)$ the *differential system of a deformation*.

Remark 5.2. The integral manifolds of $(\mathcal{I}, \alpha^1 \wedge \alpha^2)$ are 2-dimensional immersed surfaces $([A], q, p, r, w) : M \to P$ such that

- $f = [A_0 \wedge A_1] : M \to \Lambda$ is a nondegenerate Legendre immersion.
- $[A] : M \to G/\mathbb{Z}_2$ is the canonical frame along M.
- $q, p, r : M \to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ are the invariant functions of f.
- $w : M \to \mathbb{R}^3$ is a parallel section of the σ-connection of f.

Thus, the deformations of a nondegenerate Legendre immersion may be regarded as the integral manifolds of the differential system $(\mathcal{I}, \alpha^1 \wedge \alpha^2)$.

From the Maurer–Cartan equations we obtain the *quadratic equations* of the system, which are (modulo $\mathcal{I}$)

$$(5.5) \qquad d\alpha^1 \equiv -q_2\alpha^1 \wedge \alpha^2, \quad d\alpha^2 \equiv -q_1\alpha^1 \wedge \alpha^2,$$

$$(5.6) \qquad d\eta^1 \equiv \cdots \equiv d\eta^8 \equiv 0,$$

$$(5.7) \qquad \begin{cases} d\eta^9 & \equiv \ 2dq_1 \wedge \alpha^1 - dq_2 \wedge \alpha^2 + (-1 + p_2 - q_1q_2)\alpha^1 \wedge \alpha^2, \\ d\eta^{10} & \equiv \ dq_1 \wedge \alpha^1 - 2dq_2 \wedge \alpha^2 + (1 - p_1 + q_1q_2)\alpha^1 \wedge \alpha^2, \end{cases}$$

$$(5.8) \qquad \begin{cases} d\eta^{11} & \equiv \ -dr_1 \wedge \alpha^1 - dp_2 \wedge \alpha^2 + (2r_1q_2 + 3q_1p_2)\alpha^1 \wedge \alpha^2, \\ d\eta^{12} & \equiv \ -dp_1 \wedge \alpha^1 - dr_2 \wedge \alpha^2 + (3p_1q_2 + 2r_2q_1)\alpha^1 \wedge \alpha^2, \\ d\eta^{13} & \equiv \ dr_2 \wedge \alpha^1 - dr_1 \wedge \alpha^2 + 4(q_1r_1 - q_2r_2)\alpha^1 \wedge \alpha^2, \end{cases}$$

$$(5.9) \qquad d\eta^{14} \equiv -2w_1 d\eta^9, \quad d\eta^{15} \equiv -2w_2 d\eta^{10}$$

$$(5.10) \quad d\eta^{16} \equiv -w_3(d\eta^9 + d\eta^{10}) + 2w_1 dp_2 \wedge \alpha^2 - 2w_2 dp_1 \wedge \alpha^1 + 3w_3(p_2 - p_1)\alpha^1 \wedge \alpha^2.$$

From this, we see that the differential ideal $\mathcal{I}$ is algebraically generated by the 1-forms $\eta^1, \ldots, \eta^{16}$ and by the differential 2-forms

$$(5.11) \qquad \begin{cases} \Omega^1 & = \ 2dq_1 \wedge \alpha^1 - dq_2 \wedge \alpha^2 + (-1 + p_2 - q_1q_2)\alpha^1 \wedge \alpha^2, \\ \Omega^2 & = \ dq_1 \wedge \alpha^1 - 2dq_2 \wedge \alpha^2 + (1 - p_1 + q_1q_2)\alpha^1 \wedge \alpha^2, \\ \Omega^3 & = \ dr_1 \wedge \alpha^1 + dp_2 \wedge \alpha^2 - (2r_1q_2 + 3q_1p_2)\alpha^1 \wedge \alpha^2, \\ \Omega^4 & = \ dp_1 \wedge \alpha^1 + dr_2 \wedge \alpha^2 - (3p_1q_2 + 2r_2q_1)\alpha^1 \wedge \alpha^2, \\ \Omega^5 & = \ dr_2 \wedge \alpha^1 - dr_1 \wedge \alpha^2 + 4(q_1r_1 - q_2r_2)\alpha^1 \wedge \alpha^2, \\ \Omega^6 & = \ 2w_1 dp_2 \wedge \alpha^2 - 2w_2 dp_1 \wedge \alpha^1 + 3w_3(p_2 - p_1)\alpha^1 \wedge \alpha^2. \end{cases}$$

Remark 5.3. Notice that the differential system $(\mathcal{I}, \alpha^1 \wedge \alpha^2)$ is quasi-linear.

To discuss the involutivity of the system we compute the polar spaces of 1-dimensional integral elements. On P, we consider the coframe

$$(5.12) \qquad (\alpha^1, \alpha^2, \eta^1, \ldots, \eta^{16}, dq_1, dq_2, dp_1, dp_2, dr_1, dr_2)$$

and its dual frame field

$$(5.13) \qquad \left(\frac{\partial}{\partial \alpha^1}, \frac{\partial}{\partial \alpha^2}, \frac{\partial}{\partial \eta^1}, \ldots, \frac{\partial}{\partial \eta^{16}}, \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_1} \right).$$

The 1-dimensional integral elements E of the system are of the form

$$(5.14) \qquad E = [V(a, b, c, d)], \quad V(a, b, c, d) = a_j \frac{\partial}{\partial \alpha^j} + b_j \frac{\partial}{\partial q_j} + c_j \frac{\partial}{\partial p_j} + d_j \frac{\partial}{\partial r_j}.$$

Thus, the manifold of 1-dimensional integral elements $\mathcal{V}_1 \cong P \times \mathbb{RP}^7$. A 1-dimensional integral element is admissible if and only if $(a_1)^2 + (a_2)^2 \neq 0$. The polar equations of a given $E \in \mathcal{V}_1$ are

$$(5.15) \qquad \qquad \eta^\alpha = 0, \quad \alpha = 1, \ldots, 16$$

and

$$(5.16) \qquad \qquad i_V \Omega^\beta = 0, \quad \beta = 1, \ldots, 6,$$

which read

$$2a_1 dq_1 - a_2 dq_2 = [a_2(1 - p_2 + q_1 q_2) + 2b_1)]\alpha^1 + [a_1(p_2 - q_1 q_2 - 1) - b_2]\alpha^2,$$

$$a_1 dq_1 - 2a_2 dq_2 = [b_1 - a_2(1 - p_1 + q_1 q_2)]\alpha^1 + [a_1(1 - p_1 + q_1 q_2) - 2b_2]\alpha^2,$$

$$a_1 dr_1 + a_2 dp_2 = [d_1 + a_2(2r_1 q_2 + 3q_1 p_2)]\alpha^1 + [c_2 - a_1(2r_1 q_2 + 3q_1 p_2)]\alpha^2,$$

$$a_1 dp_1 + a_2 dr_2 = [c_1 + a_2(3p_1 q_2 + 2r_2 q_1)]\alpha^1 + [d_2 - a_1(3p_1 q_2 + 2r_2 q_1)]\alpha^2,$$

$$a_2 dr_1 - a_1 dr_2 = [4a_2(q_1 r_1 - q_2 r_2) - d_2]\alpha^1 + [d_1 - 4a_1(q_1 r_1 - q_2 r_2)]\alpha^2,$$

$$w_2 a_1 dp_1 - w_1 a_2 dp_2 = [w_2 c_1 - \frac{3}{2} a_2 w_3 (p_1 - p_2)]\alpha^1 + [\frac{3}{2} a_1 w_3 (p_1 - p_2) - w_1 c_2]\alpha^2.$$

Therefore, if $a_1 a_2 (w_1(a_1)^2 - w_2(a_2)^2) \neq 0$, the polar equations are linearly independent and the dimension of the polar space $H(E)$ of E is 2. Thus $H(E)$ is the only 2-dimensional integral element that contains E. This shows that the system is in involution and that the general integral submanifolds depend on six functions in one variable.

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